Assignment 3

Exercise 1

The goal of this exercise is to mimic the construction of Brownian motion done in the lectures to construct the Poisson process, which is a much simpler yet important process. Recall that Γ follows a Poisson distribution $P(\lambda)$ of parameter $\lambda > 0$ if $\mathbb{P}(\Gamma = k) = \mathrm{e}^{-\lambda}\lambda^k/k!$ for all $k \in \mathbb{N}$. We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here are some simple questions concerning Poisson random variables.

1) Let $\Gamma \sim P(\lambda)$ and $\Gamma' \sim P(\lambda')$, for $(\lambda, \lambda') \in (0, +\infty)^2$ be $\mathbb{P}$-independent. Show that $\Gamma + \Gamma' \sim P(\lambda + \lambda')$.
2) Suppose that $\Gamma \sim P(\lambda)$ and $p \in (0, 1)$. Let $(X_i)_{i \in \mathbb{N}^\star}$ be $\mathbb{P}$–i.i.d. random variables $\mathbb{P}$-independent of Γ with $\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = 0] = p$, $i \in \mathbb{N}^\star$, and define $\Gamma_p := \sum_{i=1}^{\Gamma} X_i$ and $\Gamma_{1-p} := \sum_{i=1}^{\Gamma}(1 - X_i)$. Show that $\Gamma_p \sim P(p\lambda)$, $\Gamma_{1-p} \sim P((1 - p)\lambda)$ and that Γ_p and Γ_{1-p} are $\mathbb{P}$-independent.
3) Determine the characteristic function of $P(\lambda)$.

We are now going to construct a continuous-time process $N := (N_t)_{t \in \mathbb{R}_+}$ with values in $\mathbb{N} \cup \{+\infty\}$ satisfying the following properties (N is called a *Poisson process* of rate 1):

- $N_0 = 0$, $\mathbb{P}$–a.s.;
- $N_t \sim P(t)$, for all $t > 0$;
- N has $\mathbb{P}$-independent and stationary increments, that is for all $n \in \mathbb{N}^\star$ and any $0 \leq t_0 < t_1 < \cdots < t_n$, we have that $(N_{t_i} - N_{t_{i-1}} \colon i \in \{1, \ldots, n\})$ are $\mathbb{P}$-independent and for any $0 \leq s < t$, $N_t - N_s$ and N_{t-s} have the same law;
- $t \longmapsto N_t$ is right-continuous and non-decreasing.

The goal now is to construct such a process using a countable collection of $\mathbb{P}$–i.i.d. Poisson random variables with parameter 1 and an independent countable number of i.i.d. Bernoulli random variables with parameter 1/2.

4) Show iteratively that we can construct a process $(N'_t)_{t \in D_n}$ satisfying the first three properties above where for any $n \in \mathbb{N}$, $D_n := 2^{-n}\mathbb{N}$. Check that $t \longmapsto N'_t$ defined on $\cup_{n \in \mathbb{N}} D_n$ is $\mathbb{P}$–a.s. non-decreasing.
5) Now define $N_t := \inf\{N'_s \colon s > t,\ s \in \cup_{n \in \mathbb{N}} D_n\}$ for $t \geq 0$. Show that N is a Poisson process.

We now give another construction of a Poisson process. Recall that we say that a random variable τ has an exponential distribution with parameter $\lambda \geq 0$ if its law has a density with respect to lebesgue measure given by $\lambda \mathrm{e}^{-\lambda t}\mathbf{1}_{(0,\infty)}(t)$, $t \in \mathbb{R}$. Let $(\tau_i)_{i \in \mathbb{N}^\star}$ be $\mathbb{P}$–i.i.d. exponentially distributed random variables with parameter 1, and set $N_t := \sup\{k \geq 0 \colon \tau_1 + \cdots + \tau_k \leq t\}$, $t \geq 0$.

6) Show that $N_0 = 0$, $\mathbb{P}$–a.s., and that $t \longmapsto N_t$ is right-continuous and non-decreasing.
7) Show for any $1 \leq i \leq j$ (by induction on $j - i$ or otherwise) that the law of $\tau_{[i,j]} := \tau_i + \cdots + \tau_j$ is has a density with respect to Lebesgue measure given by $t^{j-i}\mathrm{e}^{-t}/(j - i)!\mathbf{1}_{(0,\infty)}(t)$ (this is a Gamma distribution).
8) By explicit computation show that N is a Poisson process.

Exercise 2

We now use the Poisson process to construct some more complicated processes with independent stationary increments, that jump at a random dense set of times. Let $(N^{(n)})_{n \in \mathbb{Z}}$ be a sequence of i.i.d. Poisson processes and define

$$Y_t := \sum_{n=0}^{\infty} 4^{-n} N^{(n)}_{3^n t}, \text{ and } Z_t := \sum_{n \in \mathbb{Z}} 4^{-n} N^{(n)}_{3^n t},\ t \geq 0.$$

Answer the questions below.

1) Compute $\mathbb{E}^{\mathbb{P}}[Y_t]$ for $t \geq 0$. Show that $Y_t < +\infty$, $\mathbb{P}$–a.s. for $t \geq 0$ and that Y has $\mathbb{P}$-independent and stationary increments.

2) Fix $t \geq 0$. Show that $\mathbb{P}$–a.s., Y is continuous at t.

3) Show that $\mathbb{P}$–almost surely, for all intervals $(a, b) \subset [0, +\infty)$, Y is not continuous on (a, b). Show that $\mathbb{P}$–a.s., Y is increasing on $[0, +\infty)$.

4) What can you say about $\mathbb{E}^{\mathbb{P}}[Z_t]$?

5) Fix $T > 0$. Show that $\mathbb{P}$–a.s., there exists $n_0 \in \mathbb{N}$ such that $N^{(-n)}_{3^{-n}T} = 0$ for all $n \geq n_0$. Deduce that the sum in the definition of Z_t converges $\mathbb{P}$–a.s.

6) Show that Z and $(4Z_{t/3} : t \geq 0)$ have the same law.

The processes Y and Z we constructed above are examples of non-trivial subordinators (*i.e.* non-decreasing Lévy processes).